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Stress singularity at the vertex of homogeneous and composite cones for different boundary conditions☆

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ABSTRACT

A method of constructing singular solutions for conical bodies is proposed and versions of its numerical realization based on the finite-elements methods are considered. The results of numerical experiments illustrating the effectiveness and reliability of the proposed method are presented, as well as the results of a calculation of the eigenvalues, which define the form of the stress singularity for homogeneous and composite circular and non-circular cones for different boundary conditions.

It was shown in Ref. 1, that for the equations of the linear theory of elasticity in the neighbourhood of corner points the following asymptotic representation exists

$$\tau \sim \sum_{k=1} f_k r^{\lambda_k - 1}, \quad r \to 0, \quad c < \mathrm{Re}\lambda_1 < \mathrm{Re}\lambda_2 < \ldots < \mathrm{Re}\lambda_k < \ldots$$

or a more complex representation with logarithmic components in the case of multiple points of the spectrum λ_k. Here r is the distance to the corner point and f_k are the angular distribution functions of the stress field τ in the neighbourhood of the corner point, which, in the plane case, depend on one polar angular variable φ, $c = 0$, and in the three-dimensional case depend on two spherical coordinates φ, θ, $c = -1/2$.

In two-dimensional problems, an investigation of the singularity of the stresses involves an analysis of the stress state at the vertex of plane wedges. After almost half a century history of the solution of these problems, almost all possible versions of wedge-shaped bodies have been considered: homogeneous, composite, isotropic, anisotropic, different boundary conditions, etc.

For three-dimensional problems we can distinguish two classes of objects: 1) the edge of a three-dimensional wedge (the edge is not necessarily rectilinear, and the aperture angle of the wedge may vary along the edge), and 2) the vertex of a polyhedral wedge or the vertex of a cone. Interest in problems of the first class have been exhausted by the results of a number of papers, including one of the first,[2] in which it was shown that the solution of the plane and antiplane problems for wedges, obtained in planes perpendicular to the edge of a three-dimensional wedge, determines the form of the stress singularity at points of the edge through which the corresponding plane passes.

A number of papers devoted to an investigation of the stress singularity at the vertex of a polyhedral wedge or at the vertex of a cone have appeared in recent years. Different versions of numerical methods have mainly been used to solve these problems, such as the finite-elements method,[3,4] the boundary-elements method,[5,6] or by applying a Mellin transformation to the initial two-dimensional boundary integral equations.[7,8] One of the exceptions is Ref. 9, in which, using an analytical solution, the stress singularity at the vertex of a homogeneous circular cone was investigated for homogeneous boundary conditions on the side surface. Considering the publications devoted to an analysis of the stress singularity in three-dimensional bodies, the relatively small volume of numerical results obtained (compared with the two-dimensional problems) is noteworthy.

A method was described in Ref. 10, which enables data to be obtained on the nature of the stress singularity at the vertex of a cone with an elliptic base for homogeneous boundary conditions. As an extension of these investigations, we present below numerical methods which enable one to obtain new results for different kinds of cones; in particular, we present new results for homogeneous and composite circular and non-circular cones for homogeneous and mixed boundary conditions.

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1. Formulation of the problem

Consider a semi-infinite (circular or non-circular) cone, the vertex of which coincides with the centre of spherical coordinates (r, θ, φ) while the base is perpendicular to the $\theta = 0$ axis. To analyse the nature of the stress singularity we will construct eigensolutions, which are identical in form with the asymptotic representation of the solution,[1]

$$u_k(r, \theta, \varphi) = r^{\lambda}\xi_k(\theta, \varphi), \quad k = 1, 2, 3 \tag{1.1}$$

and which, in the region considered, satisfy the equilibrium equations

$$\frac{1}{1-2\nu}\operatorname{grad}\operatorname{div}\mathbf{u} + \nabla^2\mathbf{u} = 0 \tag{1.2}$$

and homogeneous boundary conditions on the side surface of the cone, where either zero displacements

$$\mathbf{u} = 0 \tag{1.3}$$

or zero stresses

$$\frac{\nu}{1-2\nu}\mathbf{n}\operatorname{div}\mathbf{u} + \mathbf{n}\cdot\nabla\mathbf{u} + \frac{1}{2}\mathbf{n}\times\operatorname{rot}\mathbf{u} = 0 \tag{1.4}$$

may be specified. Here $\mathbf{u}$ is the displacement vector, $\mathbf{n}$ is the unit vector of the outward normal and ν is Poisson's ratio.

The version of the boundary conditions corresponding to the conditions of ideal slippage on the side surface may be of interest. These conditions have the form

$$u_\theta = 0, \quad \tau_{r\theta} = 0, \quad \tau_{\varphi\theta} = 0 \tag{1.5}$$

Mixed boundary conditions, when conditions (1.3) are specified when $0 \le \varphi \le \varphi_1$ and conditions (1.4) when $\varphi_1 \le \varphi \le 2\pi$, may be assigned on the cone side surface.

In addition to a solid cone, a hollow cone, having two side surfaces, may be the subject of investigations. For a circular cone, the region occupied by the body is defined as follows: $0 \le r \le \infty$, $0 \le \varphi \le 2\pi$, $\theta_1 \le \theta \le \theta_2$ ($\theta_1 = 0$ corresponds to a solid cone). In this case, it is possible to specify one of the versions of boundary conditions (1.3)–(1.5) on the side surfaces.

To construct the eigensolutions, substituting expressions (1.1) into Eq. (1.2), we obtain a system of partial differential equations in the functions $\xi_k(\theta, \varphi)$ and the parameter λ

$$\begin{aligned}
L_1(\lambda, \xi_k) &= 2(1-\nu)(k_1-2)\xi_1 + k_2\left(\xi_2\operatorname{ctg}\theta + \xi_{2\theta} + \frac{1}{\sin\theta}\xi_{3\varphi}\right) + \\
&+ (1-2\nu)\left(\operatorname{ctg}\theta\xi_{1\theta} + \xi_{1\theta\theta} + \frac{1}{\sin^2\theta}\xi_{1\varphi\varphi}\right) = 0 \\
L_2(\lambda, \xi_k) &= \left[(1-2\nu)k_1 - \frac{2(1-\nu)}{\sin^2\theta}\right]\xi_2 + k_3\xi_{1\theta} - (3-4\nu)\frac{\operatorname{ctg}\theta}{\sin\theta}\xi_{3\varphi} + \\
&+ \frac{1-2\nu}{\sin^2\theta}\xi_{2\varphi\varphi} + \frac{1}{\sin\theta}\xi_{3\theta\varphi} + 2(1-\nu)(\operatorname{ctg}\theta\xi_{2\theta} + \xi_{2\theta\theta}) = 0 \\
L_3(\lambda, \xi_k) &= (1-2\nu)\left(k_1 - \frac{1}{\sin^2\theta}\right)\xi_3 + k_3\frac{1}{\sin\theta}\xi_{1\varphi} + (3-4\nu)\frac{\operatorname{ctg}\theta}{\sin\theta}\xi_{2\varphi} + \\
&+ \frac{2(1-\nu)}{\sin^2\theta}\xi_{3\varphi\varphi} + \frac{1}{\sin\theta}\xi_{2\theta\varphi} + (1-2\nu)(\operatorname{ctg}\theta\xi_{3\theta} + \xi_{3\theta\theta}) = 0
\end{aligned} \tag{1.6}$$

Here

$$k_1 = \lambda^2 + \lambda, \quad k_2 = \lambda - 3 + 4\nu, \quad k_3 = \lambda + 4 - 4\nu$$

$$\xi_{k\theta} = \partial\xi_k/\partial\theta, \quad \xi_{k\varphi} = \partial\xi_k/\partial\varphi, \quad \xi_{k\theta\theta} = \partial^2\xi_k/\partial\theta^2 \text{ etc.}$$

Boundary conditions (1.3) and (1.4), taking representation (1.1) into account, can be converted respectively to the form

$$M_1(\lambda, \xi_k) \equiv \xi_1 = 0, \quad M_2(\lambda, \xi_k) \equiv \xi_2 = 0, \quad M_3(\lambda, \xi_k) \equiv \xi_3 = 0 \tag{1.7}$$

$$\begin{aligned}
M_1(\lambda, \xi_k) &\equiv \xi_{1\theta} + \xi_2(\lambda - 1) = 0 \\
M_2(\lambda, \xi_k) &\equiv (1-\nu)\xi_{2\theta} + (1+\nu\lambda)\xi_1 + \nu\operatorname{ctg}\theta\xi_2 + \frac{\nu}{\sin\theta}\xi_{3\varphi} = 0 \\
M_3(\lambda, \xi_k) &\equiv \xi_{3\theta} + \frac{1}{\sin\theta}\xi_{2\varphi} - \operatorname{ctg}\theta\xi_3 = 0
\end{aligned} \tag{1.8}$$

Here L_k and M_k are differential operators.

In addition to a homogeneous cone, one can also investigate a composite cone, for example, a circular cone occupying the region $V=V_1+V_2$, where the subregion V_1 (subregion V_2) is made of a material with a shear modulus G_1 (G_2) and Poisson's ratio ν_1 (ν_2) and its geometry is given by the relations $0\le r\le\infty$, $0\le\varphi\le 2\pi$, $\theta_1\le\theta\le\theta_2$ ($\theta_2\le\theta\le\theta_3$). In special cases θ_1 and θ_3 can be equal to 0 and π respectively.

When considering a composite cone, the eigensolutions (1.1) must satisfy equilibrium Eq. (1.2) in each of the subregions V_1 and V_2. The equilibrium equations will differ solely in the values of the elastic constants of the material. Then, on the surfaces $\theta=\theta_1$ ($\theta\ne 0$) and $\theta=\theta_3$ ($\theta\ne\pi$) one of the versions of boundary conditions (1.3), (1.4) or (1.5) can be specified, while on the contact surface ideal layer bending conditions may be specified, namely,

$$u_r^{(1)} = u_r^{(2)},\quad u_\varphi^{(1)} = u_\varphi^{(2)},\quad u_\theta^{(1)} = u_\theta^{(2)};\quad \sigma_\theta^{(1)} = \sigma_\theta^{(2)},\quad \tau_{r\theta}^{(1)} = \tau_{r\theta}^{(2)},\quad \tau_{\varphi\theta}^{(1)} = \tau_{\varphi\theta}^{(2)} \tag{1.9}$$

or ideal slippage conditions

$$u_\theta^{(1)} = u_\theta^{(2)};\quad \sigma_\theta^{(1)} = \sigma_\theta^{(2)},\quad \tau_{r\theta}^{(1)} = \tau_{r\theta}^{(2)} = \tau_{\varphi\theta}^{(1)} = \tau_{\varphi\theta}^{(2)} = 0 \tag{1.10}$$

Before the stage of obtaining Eqs. (1.6) with boundary conditions of the form (1.7) or (1.8), the scheme for investigating the stress singularity is similar to the approach proposed previously for two-dimensional problems[11] or for a circular cone.[12] This is the furthest we can go in the analogy when solving two-dimensional and three-dimensional problems, since the construction of an analytical solution of Eq. (1.6) for many of the problems considered is problematic.

2. A method of calculating the indicators of stress singularity

We propose to solve the problem in question as follows. We will write Eq. (1.6) in a weak form,[13] for which we multiply them by the corresponding variations $\delta\xi_k(\theta,\varphi)$ and integrate over the area S, cut off by the cone on a sphere. We obtain

$$\int_S\left[\sum_{k=1}^{3} L_k(\lambda,\xi_1,\xi_2,\xi_3)\delta\xi_k(\theta,\varphi)\right]dS = 0 \tag{2.1}$$

We will use the finite element method to solve Eq. (2.1). The finite-element realization of these equations is fairly complex, since it is necessary to use two-dimensional elements, which, together with the continuity of the functions ξ_k, ensure continuity of their first derivatives. Without going into a detailed analysis, we note that there are no effective procedures for constructing such elements in the finite element method. In this connection, carrying out identity transformations in order to reduce the order of the derivative functions of the solutions in Eq. (2.1) and taking boundary conditions (1.8) into account, we obtain the equation

$$\begin{aligned}
&\iint_S\Big\{[2(1-\nu)(k_1-2)\sin\theta\xi_1 + k_2(\cos\theta\xi_2+\sin\theta\xi_{2\theta}+\xi_{3\varphi})]\delta\xi_1 - \\
&-(1-2\nu)\left(\sin\theta\xi_{1\theta}\delta\xi_{1\theta}+\frac{1}{\sin\theta}\xi_{1\varphi}\delta\xi_{1\varphi}\right)+\left[(1-2\nu)k_1\sin\theta\xi_2-\frac{2(1-\nu)}{\sin\theta}\xi_2+\right.\\
&\left.+k_3\sin\theta\xi_{1\theta}-(3-4\nu)\operatorname{ctg}\theta\xi_{3\varphi}\right]\delta\xi_2-2(1-\nu)\sin\theta\xi_{2\theta}\delta\xi_{2\theta}-\\
&-2\nu\xi_{3\varphi}\delta\xi_{2\theta}-(1-2\nu)\left(\frac{1}{\sin\theta}\xi_{2\varphi}\delta\xi_{2\varphi}+\xi_{3\theta}\delta\xi_{2\varphi}\right)+\left[(1-2\nu)k_1\sin\theta\xi_3-\right.\\
&\left.-\frac{1-2\nu}{\sin\theta}\xi_3+k_3\xi_{1\varphi}+(3-4\nu)\operatorname{ctg}\theta\xi_{2\varphi}\right]\delta\xi_3-\frac{2(1-\nu)}{\sin\theta}\xi_{3\varphi}\delta\xi_{3\varphi}-\\
&-2\nu\xi_{2\theta}\delta\xi_{3\varphi}-(1-2\nu)(\sin\theta\xi_{3\theta}\delta\xi_{3\theta}+\xi_{2\varphi}\delta\xi_{3\theta})\Big\}d\theta d\varphi+\\
&+\int_l\{(1-2\nu)(1-\lambda)\sin\theta\xi_2\delta\xi_1-2[(1+\nu\lambda)\sin\theta\xi_1+\nu\cos\theta\xi_2]\delta\xi_2+\\
&+(1-2\nu)\cos\theta\xi_3\delta\xi_3\}dl = 0
\end{aligned} \tag{2.2}$$

where l is the boundary of the surface S on which the stresses are specified.

The reduction in the order of the derivatives enables us to use finite elements, which ensure only continuity of the functions ξ_k. We used triangular elements with Lagrange approximation of the linear polynomials of the functions ξ_k as the finite elements.

For a numerical analysis of circular conical bodies and for unmixed boundary conditions on the side conical surfaces we can represent the functions $\xi_k(\theta, \varphi)$ in the form of a Fourier series in the circumferential coordinate φ

$$\xi_1 = \sum_{n=0}^{\infty} \beta_1^{(n)}(\theta)\cos n\varphi, \quad \xi_2 = \sum_{n=0}^{\infty} \beta_2^{(n)}(\theta)\cos n\varphi, \quad \xi_3 = \sum_{n=0}^{\infty} \beta_3^{(n)}(\theta)\sin n\varphi \tag{2.3}$$

Taking expansion (2.3) into account, we can write Eqs. (2.1) and boundary conditions (1.7) and (1.8) for each of the harmonics of the Fourier series in the following form (the prime denotes a derivative with respect to θ and the superscript (n) on β_1, β_2 and β_3 is omitted)

$$\begin{aligned}
&\int_{\theta_1}^{\theta_2} \{[2(1-\nu)(k_1-2)\sin^2\theta\beta_1 + k_2(\cos\theta\sin\theta\beta_2 + \sin^2\theta\beta_2' + n\sin\theta\beta_3) + \\
&+ (1-2\nu)(\cos\theta\sin\theta\beta_1' + \sin^2\theta\beta_1'' - n^2\beta_1)]\delta\xi_1 + [(1-2\nu)k_1\sin^2\theta\beta_2 + \\
&+ k_3\sin^2\theta\beta_1' - 2(1-\nu)\beta_2 + n\sin\theta\beta_3' - (1-2\nu)n^2\beta_2 - (3-4\nu)n\cos\theta\beta_3 + \\
&+ 2(1-\nu)(\cos\theta\sin\theta\beta_2' + \sin^2\theta\beta_2'')]\delta\xi_2 + [(1-2\nu)k_1\sin^2\theta\beta_3 - k_3 n\sin\theta\beta_1 - \\
&- (3-4\nu)n\cos\theta\beta_2 - n\sin\theta\beta_2' - 2(1-\nu)n^2\beta_3 - (1-2\nu)\beta_3 + \\
&+ (1-2\nu)(\cos\theta\sin\theta\beta_3' + \sin^2\theta\beta_3'')]\delta\xi_3\}d\theta = 0
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&M_1(\lambda, \beta_k) \equiv \beta_1' + \beta_2(\lambda - 1) = 0 \\
&M_2(\lambda, \beta_k) \equiv (1-\nu)\beta_2' + (1+\nu\lambda)\beta_1 + \nu\,\mathrm{ctg}\,\theta\beta_2 + \frac{\nu n}{\sin\theta}\beta_3 = 0 \\
&M_3(\lambda, \beta_k) \equiv \beta_3' - \frac{n}{\sin\theta}\beta_2 - \mathrm{ctg}\,\theta\beta_3 = 0
\end{aligned} \tag{2.5}$$

The use of expansion (2.3) enables us, in the numerical realization, to transfer from the solution of a two-dimensional problem to a set of individual one-dimensional problems for each of the harmonics of the Fourier series. In the one-dimensional forms of the finite element method, unlike two-dimensional problems, there are approved finite elements, which enable one to ensure continuity of the approximating functions and their first derivatives between neighbouring elements, and hence in this case a finite-element realization of Eq. (2.4) is immediately possible. We used as finite elements one-dimensional binodal elements with approximations in the element of the functions $\beta_i^{(n)}(\theta)$ by a cubic polynomial, defined by the values of the function and its derivatives $d\beta_i^{(n)}/d\theta$ at the ends of the section (a one-dimensional element).

As in the two-dimensional version, the procedure for reducing the orders of the derivatives in Eq. (2.4) can be employed. Then, when using the finite element method to solve these equations, one-dimensional elements can be used, which ensure only continuity of the approximated functions, in particular, the one-dimensional binodal elements with linear approximation of the functions $\beta_i^{(n)}(\theta)$.

3. Numerical realization and confirmation of the method

The Bubnov procedure in combination with the finite element method reduces the problem to seeking eigenvalues and eigenvectors of an algebraic asymmetric matrix having a strip structure. To solve the resultant algebraic problem of finding complex eigenvalues, we employed an algorithm based on the use of Müller's method and the argument principle, which enabled acceptable numerical results to be obtained.

The reliability and effectiveness of the proposed method, and the algorithm for its numerical realisation can be confirmed by two versions of numerical experiment. The first of these is related to the possibility of comparing numerical and analytical results[9] for a homogeneous solid circular cone ($0 \le r \le \infty$, $0 \le \varphi \le 2\pi$, $0 \le \theta \le \theta_2$). In a solid cone a stress singularity may occur for zero, first and second harmonics of the Fourier series.

In Fig. 1 we show the results for the real parts of the eigenvalue $\lambda_k < 1$ for boundary conditions on the side surface (1.3) (a) and for boundary conditions (1.4) (b) for $\nu = 0.3$. The results were obtained in the two-dimensional version of the numerical realization (Eq. (2.2)) and the one-dimensional version (Eq. (2.4)) for uniform discretization of the region considered by corresponding finite elements. The difference between the numerical and analytical results in the two-dimensional version for a number of nodal variables $\sim 10^3$ amounts to less than one percent.

The other numerical experiment involved an analysis of the convergence of the numerical method from the degree of discretization of the calculation region, as, for example, in Fig. 2, where we show the numerical solution (the continuous curve) as a function of the number of nodal variables N and the analytical results (the dash-dot curve) for $\theta_2 = 2\pi/3$ and $\nu = 0.3$. The results of similar experiments demonstrate not only the convergence of the numerical procedure, but also enable us to choose the type of discretization of the calculation region, which ensures acceptable accuracy.

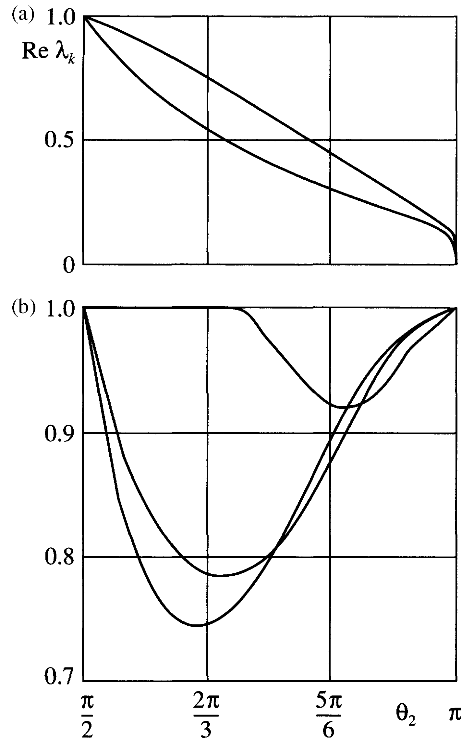

Fig. 1.

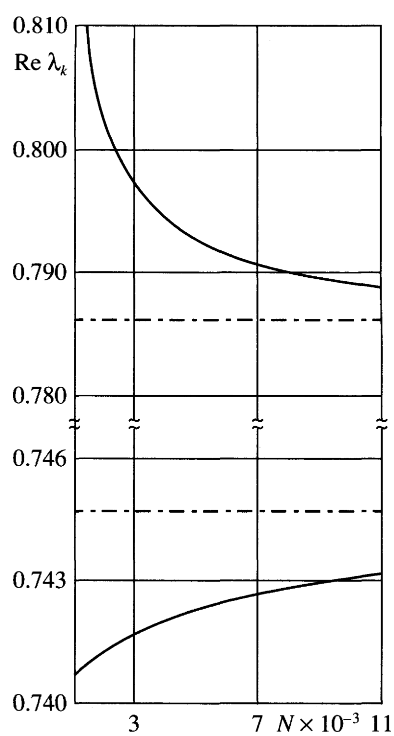

Fig. 2.

4. Results

We will present further the results of a solution of new problems. In Fig. 3 we show the real parts of the eigenvalue $\lambda_k < 1$ for a solid circular cone ($\nu = 0.3$) with boundary conditions (1.5), corresponding to the conditions of ideal slippage. In Fig. 4 we show the real parts of the eigenvalue for one of the versions of a solid circular cone ($\theta_2 = 2\pi/3$ and $\nu = 0.3$) for mixed boundary conditions on the side surface: zero displacements are specified for $0 \leq \varphi \leq \varphi_1$ and zero stresses are specified for $\varphi_1 \leq \varphi \leq 2\pi$. It should be noted

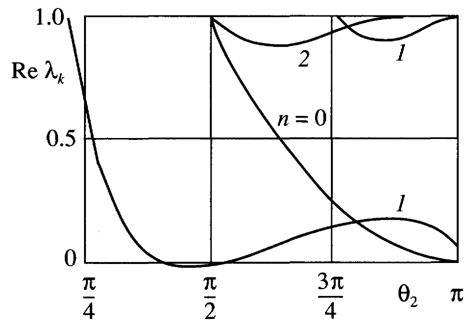

Fig. 3.

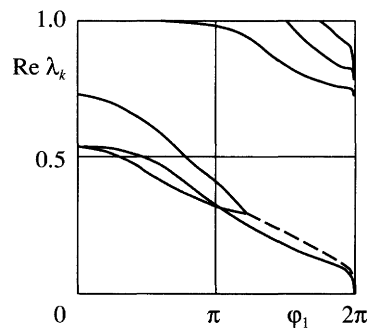

Fig. 4.

that, when solving this problem, one cannot represent the required solution in the form of a Fourier series in the angular coordinate φ. Here and henceforth the continuous curve corresponds to the real eigenvalue and the dashed curve corresponds to the complex eigenvalue.

We carried out calculations for a composite cone, which enables the effect of the ratios of the mechanical characteristics on the indicator of the stress singularity to be estimated. In Fig. 5 we show values of Re$\lambda_k < 1$ as a function of the ratio G_1/G_2 for a composite cone with boundary conditions (1.9) and $\theta_1 = 0$, $\theta_2 = \pi/3$, $\theta_3 = 2\pi/3$ and $\nu_1 = \nu_2 = 0.3$.

The method considered enables us to obtain numerical results for different cone shapes, including for a cone the base of which is an ellipse. The geometry of the boundary S (2.2), cut out by a cone on a sphere, is given by the relation

$$\mathrm{tg}\,\theta = \mathrm{tg}\,\theta_2(1/\sqrt{\cos^2\varphi + \varkappa^{-2}\sin^2\varphi}), \quad \varkappa = a/b \tag{4.1}$$

Here a and b are the semi-axes of the ellipse and $2\theta_2$ is the aperture angle of the cone in the plane passing through the vertex of the cone and the semi-axis a. In Fig. 6 we show the results of calculations of the real parts of the eigenvalues for zero stresses (a) and for zero displacements (b) on the cone side surface.

The proposed method enables one to calculate all the eigenvalues of interest. Moreover, it enables one to calculate multiple eigenvalues within the limits of the error of the numerical method. Thus, analytical results were presented in Ref.9 on the multiplicity of the eigenvalue $\lambda = 1$. In particular, the multiplicity is equal to 6 for $\theta_2 = \pi/2$ and 9 for $\theta_2 = \pi$. The method considered here enables us to obtain all the multiple eigenvalues up to the third place for a number of finite elements of the order of three thousand.

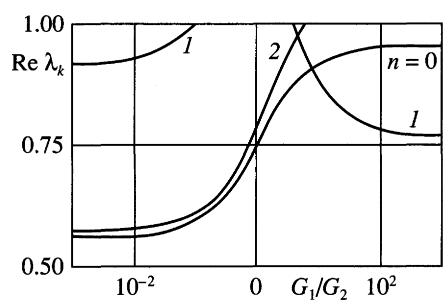

Fig. 5.

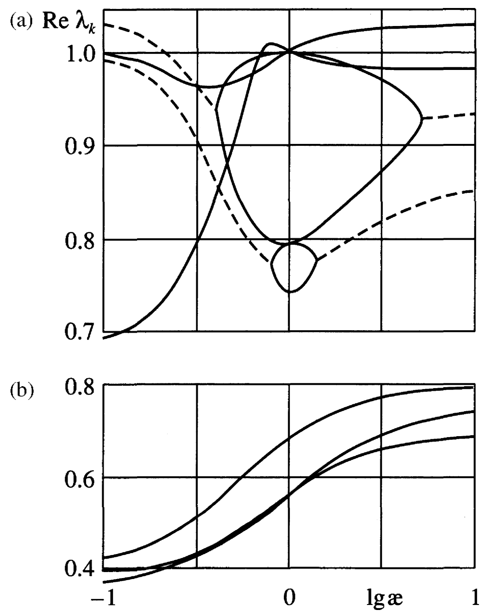

Fig. 6.

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